Basic Error Propagation Using Algebra
by
Danny R. Swain, P.S.M.

One of the main functions that any surveyor is tasked with is being an expert in measurement science, and part of this is understanding how errors propagate. Generally error propagation is taught using partial derivatives from calculus, but unfortunately most forget this type of math after they leave school. However, every surveyor I've ever known is quite good at algebra and trigonometry, therefore, it's my hope that reviewing this topic from an algebraic perspective will be beneficial to the reader.

While most will not be used herein, I'd like to spend a few minutes reviewing a few basic definitions from statistics that are normally associated with the study of error propagation:

1) Statistics are used in surveying to analyze repeated measurements and draw conclusions about the quality of said measurements and are based on the laws of probability.
2) Population – all possible measurements that could be made. Generally infinite and unknown.
3) Sample – a subset of data from the population. What we work with in surveying.
4) Arithmetic Mean (Average) – will be denoted by $\ddot{X}$ in this article
    a. $\ddot{X} = \Sigma x_i \div n$
        i. Where Σx_i = the summation of each individual measurement
        ii. n = the total number of measurements
5) Residual – the difference between the adjusted value and the measured value, or the amount a measurement is adjusted. It'll be denoted by *v* herein.
    a. $v = \ddot{X} - x_i$
6) Variance – measures the precision of the dataset and will be denoted by S^2.
    a. $S^2 = (\Sigma v^2) \div (n-1)$
7) Standard Deviation – is the square root of the variance and is used to draw conclusions about probability. It'll be denoted by S in this discussion.
    a. $S = \sqrt{S^2}$

Again, while most of the aforementioned definitions will not be used herein, it's highly encouraged that the reader spend a few minutes examining how these terms relate to this topic at the conclusion of the article.

Dr. Charles Ghilani wrote "Unknown values are often determined indirectly by making direct measurements of other quantities which are functionally related to the desired unknowns, then the unknowns are computed". We see this in computing coordinates from angles and distances; obtaining station elevations from rod readings in differential leveling; etc.

All directly measured quantities contain errors; therefore, any value computed from them contain errors as well. This process of moving errors forward is the topic of this article, error propagation.

In this discussion it's assumed that only random errors are being propagated during the procedure, and that all systematic errors have been removed prior to the error propagation taking place.

By analyzing how errors propagate, a general expression can be developed for the propagation of random errors through any function. This expression is generally expressed in the form of partial derivatives, but for this article we'll take a look at how this process can be performed using basic algebra.

This algebraic process can be most easily understood in my opinion by breaking it down into the following steps and then illustrating the process with a few basic examples:

1) Calculate the quantity as if there's no error.
2) Calculate the quantity after adding the error estimate to the first measurement. Subtract this value from the one derived in step one.
3) Calculate the quantity after adding the error estimate to the second measurement (the first measurement should no longer have the error estimate added). Subtract this value from the one derived in step one.
4) Repeat this process until each value and its corresponding error estimates have been used.
5) Calculate the total estimated error by taking the square root of the sum of the squares.
    a. Basic equation for error propagation.

Let's work through a few basic examples and tie some things together:

1) Basic equation for error propagation
    a. $\sqrt{(e_{x1}^2 + e_{x2}^2 + \ldots + e_{xn}^2)}$
        i. Basic example using measured distances with corresponding error estimates:
            1. 100.00' ±0.01'
            2. 200.00' ±0.02'
            3. 300.00' ±0.03'
                a. Total distance is 600.00' with an estimates error of:
                    i. $\sqrt{(0.01^2 + 0.02^2 + 0.03^2)} = \pm 0.04'$
                b. 600.00' ±0.04'
                    i. Using 1.96 as a multiplier allows us to develop a confidence interval at 95% (1.96 * 0.04' = ±0.08')
                        1. Thus any measurement (sample) between 599.92' and 600.08' would have a 95% chance of being from the same population as ours.
2) A rectangle measures 100.00' ±0.05' by 250.00' ±0.09', what is the area and its corresponding error estimate?
    a. (100.00')(250.00') = 25,000 ft^2 (without error)
    b. (100.00' + 0.05')(250.00') = 25,012.5 ft^2 (estimated error added to first measurement)
        i. 25,000 ft^2 - 25,012.5 ft^2 = -12.5 ft^2 (subtraction of value b from a)
    c. (100.00')(250.00' + 0.09') = 25,009 ft^2 (estimated error added to second measurement)
        i. 25,000 ft^2 - 25,009 ft^2 = -9 ft^2 (subtraction of value c from a)
    d. Propagate the total error by taking the square root of the sum of the squares
        i. $\sqrt{(12.5^2 + 9^2)} = \pm 15.4$ ft^2
            1. I just use the absolute value since the negatives will become positives after they're squared
    e. While the area equals 25,000 ft^2, it should be recognized the area contains an estimated error of ±15.4 ft^2
3) A rectangular tank measures 40.00' ±0.05' x 20.00' ±0.03' x 10.00' ±0.02', determine the volume and the estimated error.
    a. (40.00')(20.00')(10.00') = 8,000 ft^3 (without error)
    b. (40.00' + 0.05')(20.00')(10.00') = 8,010 ft^3 (estimated error added to first measurement)
        i. 8,000 ft^3 – 8,010 ft^3 = -10 ft^3 (subtraction of value b from a)
    c. (40.00')(20.00' + 0.03')(10.00') = 8,012 ft^3 (estimated error added to second measurement)

i. $8{,}000\ ft^3 – 8{,}012\ ft^3 = -12\ ft^3$ (subtraction of value c from a)

d. $(40.00')(20.00')(10.00' + 0.02') = 8{,}016\ ft^3$ (estimated error added to third measurement)

i. $8{,}000\ ft^3 – 8{,}010\ ft^3 = -16\ ft^3$ (subtraction of value d from a)

e. Propagate the total error by taking the square root of the sum of the squares

i. $\sqrt{(10^2 + 12^2 + 16^2)} = \pm 22.4\ ft^3$

f. While the volume equals 8,000 ft^3, it should be recognized the volume contains an estimated error of ±22.4 ft^3

4) Given point B has been set with respect to A along an azimuth of 40° ±30" and a distance of 100.00m ±2cm, what is the estimated error in position B with respect to A?

a. This can be solved exactly like the examples above, but I'll illustrate a shortcut that works quite well.

i. The shortcut is predicated on recognizing the following:

1. The estimated error in distance is along the line
2. The estimated error in azimuth (angle) is perpendicular to the line

a. From these, we can construct a right triangle that will allow us to convert our angular measurement into a linear one

i. (Tan 30")(100.00m) = ±0.015m

b. Propagate the distance estimated error with the angular (now linear) estimated error to determine the overall estimated error:

i. $\sqrt{(0.02^2 + 0.015^2)} = \pm 0.025m$

1. The estimated positional error in point B with respect to A is ±0.025m

a. Using 1.96 as a multiplier allows us to develop a confidence interval at 95% (1.96 * 0.025m = ±0.049m)

i. If we placed the center of a circle with a radius of 0.049m at the coordinates of our point, we'd expect 95% of other measured coordinate values (using similar measurement techniques and procedures) to fall within said circle

1. If this is the case, then we can statistically state that we're 95% confident that the individually derived coordinate pairs are from the same population, which means they are statistically equivalent

In closing, it's worth noting the following:

1) This algebraic approach will work for any situation, but can become cumbersome depending on the number of variables.
2) Understanding error propagation is essential for understanding least squares adjustments and weighting.
3) There are equations in textbooks for estimating errors in horizontal angles, zenith angles, distances, GPS vectors, etc.

a. A quick example of a GPS vector would be:

i. $S_v = \sqrt{(S_{di}^2 + Sd_{di}^2 + a^2 + (d*b\ ppm)^2)}$

1. S_v = Estimated error in GPS vector
2. S_{di} = Estimated error in centering the GPS antenna over the station (One on each end of the vector)
3. a = GPS Constant (from manufacturer)
4. d = 3d vector length

5. b = Parts per million (from manufacturer)

ii. A GPS vector determined by rapid static has the following:

1. (5mm + 1 ppm) from the manufacturer
2. 3d vector length of 10,500m
3. The two antennas were centered within 3mm

a. What is the estimated error in the vector?

i. $\sqrt{(0.003^2 + 0.003^2 + 0.005^2 + (10{,}500/1{,}000{,}000)^2)}$ = ±12mm

1. 95% = ±23.5mm